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CONDITIONS FOR FINITE SIGNAL SPEED IN RELAXATIONAL INFILTRATION

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Relaxation effects can occur in the interaction between the rock skeleton and the fluid during nonstationary liquid or gas infiltration [1-3]. The frequency dependence of the permeability has been considered repeatedly in the literature since the appearance of [4, 5]. In [6], the finite signal propagation rate was examined for a linearly elastic porous medium saturated with a compressible fluid. However, a fairly serious assumption was made [6] on the boundedness of the relaxation kernel at short times, even though it was stated that there are examples where this condition is not met.

Here I examine the finite signal speed for the case where the relaxation kernel may have singularities at short times.

I consider a homogeneous isotropic linearly-elastic porous medium saturated with a compressible fluid. I describe the possible dynamic processes by means of the [6, 7] symbols: ρ_1 the solid-phase density, ρ_2 the liquid-phase density, m porosity, l_i the solid-phase displacement vector, w_i the mean particle velocity in the liquid phase, σ_{ij} the solid-phase stress tensor, and p the pressure in the liquid phase. The Latin subscripts run through the range 1, 2, and 3, which correspond to the coordinates in the $Ox^1x^2x^3$ Cartesian system, with summation with respect to the related subscripts; t is time. All the processes are assumed to be isothermal.

In the linearly elastic formulation,

$$\sigma_{ij} = \lambda_1 e\delta_{ij} + 2\lambda_2 e_{ij}, \quad e_{ij} = \frac{1}{2}\left(\frac{\partial l_i}{\partial x^j} + \frac{\partial l_j}{\partial x^i}\right), \quad e = e_{ij}, \tag{1}$$

in which λ_α are the Lamé coefficients. Let R_i be the interaction between the liquid and solid phases. We use an expression from relaxation theory for it [6]:

$$R_i\Big|_{t_0} = \frac{\mu m^2}{k}\int_{-\infty}^{t_0} K(t_0 - t)\left(w_i - \frac{\partial l_i}{\partial t}\right)\Big|_t dt. \tag{2}$$

Here μ is the shear viscosity, k the permeability, and K(t) the relaxation kernel, which is normalized in such a way that

$$\int_0^{+\infty} K(t)\,dt = 1. \tag{3}$$

Consider the propagation of weak perturbations in this two-phase medium. Subscript 0 is used with quantities relating to the unperturbed state. Let $\rho_\alpha' = \rho_\alpha - \rho_{\alpha 0}$ ($\alpha = 1, 2$) be the perturbations in the density patterns. We use linear expressions for the pressure and porosity: $p = p_0 + c^2\rho_2'$, $m = m_0 + a_2\rho_2' - a_1\rho_1'$, with c the isothermal speed of sound.

The following are the dynamic equations for the perturbations derived from (1) and (2) in the absence of sources:

$$\frac{\partial}{\partial t}(\alpha_1\rho_1' - \alpha_2\rho_2') + \frac{\partial^2 l_i}{\partial t\,\partial x^i} = 0, \quad \frac{\partial}{\partial t}(-\beta_1\rho_1' + \beta_2\rho_2') + \frac{\partial w_i}{\partial x^i} = 0, \tag{4}$$

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$$\frac{\partial}{\partial x^i}(-\zeta_1\rho_1' + \zeta_2\rho_2') + \left(\frac{\partial^2 l_i}{\partial t^2} - \eta_1\frac{\partial^2 l_j}{\partial x^i \partial x^j} - \eta_2\Delta l_i - \nu K * \frac{\partial l_i}{\partial t}\right) - \nu K * w_i = 0,$$
$$\gamma\frac{\partial\rho_2'}{\partial x^i} - \delta K * \frac{\partial l_i}{\partial t} + \delta K * w_i + \frac{\partial w_i}{\partial t} = 0.$$

When there are sources, the corresponding terms must be inserted on the right in (4). The following symbols [6] are used in (4): * time convolution,

$$\alpha_1 = \rho_{10}^{-1} + a_1(1 - m_0)^{-1};\quad \alpha_2 = a_2(1 - m_0)^{-1};$$
$$\beta_1 = a_1 m_0^{-1};\quad \beta_2 = \rho_{20}^{-1} + a_2 m_0^{-1};\quad \eta_1 = (\lambda_{10} + \lambda_{20})\rho_{10}^{-1};\quad \eta_2 = \lambda_{20}\rho_{10}^{-1};$$
$$\zeta_1 = p_0 a_1(1 - m_0)^{-1}\rho_{10}^{-1};\quad \zeta_2 = p_0 a_2(1 - m_0)^{-1}\rho_{20}^{-1};$$
$$\nu = m_0^2\mu_0(1 - m_0)^{-1}k_0^{-1}\rho_{10}^{-1};\quad \gamma = c^2\rho_{20}^{-1};\quad \delta = m_0\mu_0 k_0^{-1}\rho_{20}^{-1}.$$

We recall the properties of the Fourier transform for the kernel $\tilde{K} = \tilde{K}(\omega)$ [6] ($\tilde{K} = \tilde{K}(\omega)$ is a complex function holomorphic in the lower complex plane). In accordance with (3), $\tilde{K}(0) = 1$. For real ω, we have the thermodynamic inequality

$$\operatorname{Re}\tilde{K}(\omega) > 0. \tag{5}$$

Consider the behavior of $\tilde{K}(\omega)$ for $|\omega| \to +\infty$, $\operatorname{Im}\omega \le 0$. The asymptote adopted in [6] was

$$\tilde{K}(\omega) = K(0)(i\omega)^{-1} + O(|\omega|^{-1}), \tag{6}$$

and corresponds to the case where K(t) is a smooth function for $t \ge 0$. We now consider the wider class of asymptotes

$$\tilde{K}(\omega) = \varkappa(i\omega)^{\varepsilon} + O(|\omega|^{\varepsilon}). \tag{7}$$

Here $\varkappa$ and ε are real numbers and $\varkappa > 0$; $-1 \le \varepsilon \le 1$. The (7) asymptote class is compatible with (5) and includes (6) as a particular case. It also covers standard exact solutions for the resistance forces acting on a body moving in a linearly-viscous liquid at a variable speed [6]. As the asymptote to $\tilde{K}(\omega)$ at high frequencies is determined by the asymptote to K(t) at short times, (7) means that the convolution operation with relaxational kernel can be written as

$$K* = \varkappa_{-\infty}D^{\varepsilon} + z,$$

where the first term is the fractional time derivative [8] and z is a pseudodifferential operator of higher-order smoothness.

It follows from (5) and (7) that (5) is obeyed throughout the lower complex half-plane.

I now consider perturbations propagating from a certain source of mass or force that acts at time $t = 0$ at the origin.

We concentrate attention on perturbations at a point in space $x_0^j = L\delta_1^j$, $L > 0$. We use the [6] method to get an expression for the Fourier transforms of the perturbations in the density, velocity, and displacement patterns:

$$y(\omega) = 2\pi i\sum_{\alpha=1}^{3}\operatorname{Res}_{n_\alpha(\omega)}\left[\frac{e^{inL}N(\omega, n)q}{P_1^2(\omega, n)P_2(\omega, n)}\right]. \tag{8}$$

Here $n_\alpha = n_\alpha(\omega)$; $\alpha = 1, 2, 3$ are the roots of the equations

$$P_1 = 0; \tag{9}$$

$$P_2 = 0, \tag{10}$$

that satisfy $\operatorname{Im} n_\alpha > 0$; while P_1 and P_2 are polynomials in the complex variable n:

$$P_1 = (-\omega + i\delta\tilde{K})n^2 + \eta_2^{-1}\omega^2(\omega - i(\nu + \delta)\tilde{K}),$$
$$P_2 = An^4 + (-\omega^2 B + i\omega C\tilde{K})n^2 + i\omega^4 - i\omega^3 D\tilde{K},$$

$$A = C_1C_4 - C_2C_3,\ \ B = C_1 + C_2,\ \ C = \nu\,(C_1 + C_4) + \delta\,(C_2 + C_3),\ \ D = \nu + \delta,$$
$$C_1 = \eta_1 + \eta_2 + (\beta_1\zeta_2 - \beta_2\zeta_1)/(\alpha_1\beta_2 - \alpha_2\beta_1),$$
$$C_2 = (\alpha_1\zeta_2 - \alpha_2\zeta_1)/(\alpha_1\beta_2 - \alpha_2\beta_1),$$
$$C_3 = \gamma\beta_1/(\alpha_1\beta_2 - \alpha_2\beta_1),\quad C_4 = \gamma\alpha_1/(\alpha_1\beta_2 - \alpha_2\beta_1).$$

In (8), $N(\omega, n)$ is a matrix homomorphically dependent on ω for $\mathrm{Im}\,\omega < 0$ and polynomially dependent on n, while q is a constant vector.

The central point in calculating the perturbation propagation speed is the proof that $n_\alpha(\omega)$ never intersects the real axis when ω varies in the lower complex half-plane, i.e., it holomorphically maps the lower complex half-plane into the upper one. If this is proved, one gets the velocities V_α for the various modes of perturbation from [6]

$$V_\alpha = \left(\lim_{\substack{|\omega| \to +\infty \\ \mathrm{Im}\,\omega < 0}} (\mathrm{Im}\, n_\alpha(\omega)/(-\mathrm{Im}\,\omega))\right), \tag{11}$$

which is a standard result for problems of that type [9, 10]. We thus show that (9) and (10) as functions of n do not have real solutions for $\mathrm{Im}\,\omega < 0$. It is readily seen that for this purpose it is sufficient to show that the functions of parameter ω for real n are

$$X_1(\omega) = \bar{K} + i\omega\,(\omega^2 - \eta_2 n^2)\,((\nu + \delta)\,\omega^2 - \delta\eta_2 n^2)^{-1},$$
$$X_2(\omega) = \bar{K} + i\omega^{-1}\,(An^4 - Bn^2\omega^2 + \omega^4)\,(D\omega^2 - Cn^2)^{-1}$$

and do not become zero for $\mathrm{Im}\,\omega < 0$. Let $n > 0$. Then on the edge Γ (Fig. 1, $\omega_0 = n(\delta\eta_2)^{1/2}(\nu + \delta)^{-1/2}$), which encompasses the lower complex half-plane, we have that the following applies on account of (5) and (7):

$$\mathrm{Re}\,X_1(\omega) > 0.$$

As $X_1(\omega)$ is holomorphic in the lower complex half-plane, that inequality is obeyed for all ω, $\mathrm{Im}\,\omega < 0$. If now we consider $X_2(\omega)$, on Γ (Fig. 1, $\omega_0 = n(C/D)^{1/2}$) we have

$$\mathrm{Re}\,X_2(\omega) > 0 \tag{12}$$

subject to the additional condition

$$AD^2 - BDC + C^2 < 0. \tag{13}$$

Because of the holomorphic behavior, (12) is obeyed for all ω, $\mathrm{Im}\,\omega < 0$.

We note that (13) is not too restrictive. For example, one can usually assume in practice that $\alpha_2 = 0$ (whence $C_2 = 0$), while δ and ν are quantities of the same order, and C_1 substantially exceeds C_3 and C_4. Those conditions are sufficient for (13) to apply.

Then we can use (11), and (7) gives a result that extends the [6] conclusions:

$$V_1 = \begin{cases} \eta_2^{1/2}, & \varepsilon < 1, \\ \eta_2^{1/2}\,(1 + \delta\varkappa)^{1/2}\,(1 + \varkappa\,(\delta + \nu))^{-1/2}, & \varepsilon = 1, \end{cases}$$

$$V_{2,3} = \begin{cases} 2^{1/2}\,(B \pm (B^2 - 4A)^{1/2})^{1/2}, & \varepsilon < 1, \\ 2^{-1/2}\,(1 + \varkappa D)^{-1/2}\,(B - \varkappa C \pm ((B - \varkappa C)^2 - 4A\,(1 + \varkappa D))^{1/2})^{1/2}, & \varepsilon = 1. \end{cases}$$

Here V_1 is the limiting propagation speed for transverse perturbations, while V_2 and V_3 are the limiting propagation speeds for perturbations of the first and second types correspondingly (see [7] for terminology).

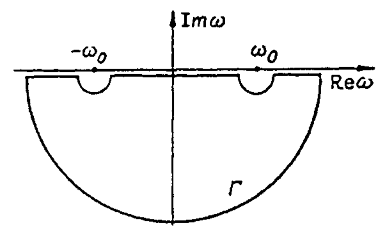

Fig. 1

When the (7) and (13) conditions apply, the signal speed is finite in relaxational infiltration. Situations can occur (ε = 1) when internal relaxation in the porous medium-saturating fluid system influences the limiting perturbation propagation speed. This substantially supplements the [6] results.

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